ON THE SYMMETRIES AND SIMILARITY SOLUTIONS OF ONE-DIMENSIONAL, NON-LINEAR THERMOELASTICITY

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Abstract

The homogeneous, one–dimensional, non–linear thermoelasticity is studied from the point of view of symmetries and similarity solutions. Special cases of free energy function and conductivity function are considered and the corresponding admitted symmetry group of transformations are derived. Also, the similarity solutions, if any, for each symmetry group are provided. Finally, the whole procedure is checked by means of obtaining the reduction of the system of partial differential equations to a system of ordinary differential equations by the insertion of the similarity solutions into them.

1 Introduction

This paper is the second part of a work concerning the symmetries of non-linear, one-dimensional, dynamical thermoelasticity. In the first part [1] the general non-homogeneous problem was concidered. In [2] some preliminary results concerning the homogeneous case are presented. The present paper intends to exhaust the homogeneous case.

By the term "non-linear", we mean non-linearity coming into the system through constitutive relations i.e., assuming a general non-linear (actually more than quadratic) free energy function. On the other hand, this is not the real full non-linear thermoelasticity because the linear relation between heat conduction and temperature field i.e., the well-known Fourier law is considered.

The concept of symmetry of a differential equation has been introduced by Sophus Lie one hundred years ago. To find out the symmetries of a differential equation means to find out the continuous group of transformations (actually they are Lie groups) under which the differential equation is invariant. Having such symmetries, one can obtain new solutions from an existing one and the so-called group-invariant solutions [3]; the well-known similarity (or self-similar) solutions is nothing but a special case of group-invariant solutions corresponding to the scaling group. Generally speaking, the more symmetries of a differential equation we know the more we know about the differential equation itself.

The fundamental ideas of S. Lie can be fruitfully coupled with concepts coming from exterior calculus [4]. According to this the main idea is based on Cartan's work by which one can obtain a geometric description of a partial differential equation in terms of closed ideals of exterior differential forms. Next, one has to find out the so-called isovector fields which in turn are defined to be the vector fields on the space of dependent and indepedent variables, over which the Lie derivatives of Cartan's exterior differential forms remain invariant. These are nothing more but the infinitesimal generators [3] of the Lie group of transformations.

Most of the researchers of the area use the so–called determining equations i.e., an overdetermined system of linear PDEs which govern the components of the isovector field, to obtain the symmetries of a differential equation. This procedure to obtain symmetries relies directly on Lie groups theory applied to the particular case of transformations group. We refer to the books of Ibragimov [5], Olver [3] and Bluman and Cole [6] for further information for the interested reader.

In Sect. 2, we summarize some results derived in [1] which are useful to the present paper. In Sect. 3, we give the isovector field for the homogeneous thermoelasticity and in Sect. 4, we examine special cases of symmetries corresponding to various cases of free energy and conductivity functions. Finally, in Sect. 5, we provide the similarity solutions arising from every non–trivial symmetry.

2 Equations of Thermoelasticity and Some Previous Results

In the first part of this work [1] we examined the system of thermoelasticity equations:

$$\frac{\partial}{\partial X}\left(\frac{\partial F}{\partial p}\right)-\frac{\partial}{\partial t}(\rho_0(X)v)=0,$$
$$\frac{\partial}{\partial X}\left(k(X)\frac{\partial \theta}{\partial X}\right)+\frac{\partial}{\partial t}\left(\theta\frac{\partial F}{\partial \theta}\right)-\frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial t}=0, \tag{1}$$

where $x(X,t)$ is the motion of the body, $p = gradx$ is the gradient of deformation, $v = \frac{\partial x}{\partial t}$ is the velocity, $\theta(X,t)$ is the absolute temperature field, $\rho(X)$ is the mass density, $k = k(X)$ is the conductivity function and $F = F(X,p,\theta)$ is the free energy function. By X is denoted the material coordinate and by t the time. The symmetries of this system are given by the components of the isovector field given by the equations [1]

$$\begin{aligned}
&\omega^1(X) = aX + c_1,\\
&\omega^2(t) = b_2t + b_3,\\
&\Omega^1(X,t,x) = a_2x + \beta_3t + \beta_4(X),\\
&\Omega^2(\theta) = \mu_2\theta,
\end{aligned} \tag{2}$$

where $a = \frac{1}{2}(b_2 - a_2 - \frac{c}{2})$ and $a_2, b_2, b_3, \beta_3, \mu_2, c, c_1$ are arbitrary constants and β_4 is an arbitrary function of X.
The free energy function F should have the form

$$F(X,p,\theta) = f(X,p)\theta^2 + \phi(X,p), \tag{3}$$

where the functions f and ϕ should fulfil the partial differential equations

$$(5a + 2\mu)f + (aX + c_1)f_X + (\beta'(X) + pb)f_p = 0, \tag{4}$$

$$5a\phi + (aX + c_1)\phi_X + (\beta'(X) + pb)\phi_p = 0, \tag{5}$$

where

$$b = -\frac{1}{2}(b_2 - 3a_2 - \frac{c}{2}), \quad \mu = \mu_2, \quad \beta(X) = \beta_4(x). \tag{6}$$

In the present paper we focus our attention on non-linear homogeneous thermoelasticity the field equations of which can be written in the form of balance equations as

$$\frac{\partial}{\partial X}(\frac{\partial F}{\partial p}) - \frac{\partial(\rho_0 v)}{\partial t} = 0 \tag{7}$$

$$\frac{\partial}{\partial X}(k(X)\frac{\partial \theta}{\partial X}) + \frac{\partial}{\partial t}(\theta\frac{\partial F}{\partial \theta}) - \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial t} = 0, \tag{8}$$

where ρ_0 is the mass density which is now considered constant throughout the body. The free energy function does not depend any more on space variable X, namely, $F = F(p, \theta)$. It is important to note that although we assume homogeneity in material properties, we do not consider constant thermal conductivity as one could expect. on the contrary, we continue to consider non-homogeneity with respect to thermal properties because as it will be apparent later accepting constant conductivity will cause cancellation of every non-trivial symmetry. Also, it is worthwhile to remark that actually we do not have a single system of partial differential equations but a class of systems depending on the particular form of free energy function F.

The main step towards the symmetries of the system is to obtain the isovector field, that is a vector field of the form

$$V = \omega^1\frac{\partial}{\partial X} + \omega^2\frac{\partial}{\partial t} + \Omega^1\frac{\partial}{\partial x} + \Omega^2\frac{\partial}{\partial \theta} \tag{9}$$

To find out the isovector field (infinitesimal generator) of the system we follow a method proposed by Suhubi [7]. This method was used by Suhubi himeself to study the similarity solutions for plane waves in hyperelastic materials [8]. Also, the author of this article used the method to study the non-homogeneous, one-dimensional problem of thermoelasticity. The interested reader can find in [1] the formulation of the problem and all technical details which are not repeated here. For the time being, the only difference with [1] is that f as well as ρ_0 do not depend on the space variable X, that means we adopt what is obtained in [1] for the isovector field, namely eqs (3.33):

$$\omega^1 = \omega^1(X), \quad \omega^2 = \omega^2(t), \quad \Omega^1 = \Omega^1(X, t, x), \quad \Omega^2 = \Omega^2(X, t, \theta).$$

Furthermore, we have to adopt equations (3.34), (3.35), (3.36) and (3.39) of [1] provided that all partial derivatives with respect to X are taken zero. An elaboration of these equations in a manner analogous to [1], will give us the following equations:

$$(-\omega^{1'}+2\dot{\omega^2}-\frac{\partial\Omega^1}{\partial x}+\frac{\partial\Omega^2}{\partial\theta})F_{p\theta}+\Omega^2F_{p\theta\theta}+(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x})F_{pp\theta}=0, \quad (10)$$

$$\frac{\partial\Omega^2}{\partial X}F_{p\theta}-\rho_0\frac{\partial^2\Omega^1}{\partial t^2}+(\frac{\partial^2\Omega^1}{\partial X^2}-p\omega^{1''}+2p\frac{\partial^2\Omega^1}{\partial X\partial x})F_{pp}=0, \quad (11)$$

$$[(k'/k)\omega^1+\dot{\omega^2}](\theta F_{\theta\theta}+F_\theta)=-\frac{\partial\Omega^2}{\partial\theta}F_\theta-\omega^{1'}F_\theta+\Omega^2(F_{\theta\theta}+\theta F_{\theta\theta\theta})$$
$$+\theta F_{\theta\theta p}(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x}), \quad (12)$$

$$k\frac{\partial^2\Omega^2}{\partial X^2}+k'\frac{\partial\Omega^2}{\partial X}+\frac{\partial\Omega^2}{\partial t}\theta F_{\theta\theta}+(\frac{\partial^2\Omega^1}{\partial X\partial t}+p\frac{\partial^2\Omega^1}{\partial x\partial t})\theta F_{\theta p}=0, \quad (13)$$

$$[(2\omega^{1'}-2\dot{\omega^2}]F_{pp}-\Omega^2F_{pp\theta}-(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x})F_{ppp}=0, \quad (14)$$

$$\Big[(k'/k)\omega^1-\omega^{1'}+\dot{\omega^2}+\frac{\partial\Omega^2}{\partial\theta}-\frac{\partial\Omega^1}{\partial x}\Big]\theta F_{\theta p}=\Omega^2(F_{\theta p}+\theta F_{\theta\theta p})+$$
$$(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x})\theta F_{\theta pp}. \quad (15)$$

Moreover, we obtain straightforwardly from eq. (3.34) and (3.35) of [1], some information concerning the components of the isovector field, namely the relations:

$$\frac{\partial^2\Omega^1}{\partial x^2}=0,\quad \frac{\partial^2\Omega^1}{\partial x\partial t}=\frac{1}{2}\ddot{\omega^2},\quad \frac{\partial^2\Omega^1}{\partial X\partial x}=0,$$
$$\frac{\partial^2\Omega^2}{\partial\theta^2}=0,\quad \frac{\partial^2\Omega^2}{\partial X\partial\theta}=\frac{1}{2}(-[(k'/k)\omega^1]'+\omega^{1''}). \quad (16)$$

Next, a detailed elaboration of these equations (see Appendix) will give the following information concerning the general form of isovector field, i.e., the symmetries of eqs. (7)–(8) as well as some constraints on the general form of F and k. Hence the isovector field will take the form:

$$\begin{aligned}
&\omega^1(X)=c_1X+c_2,\\
&\omega^2(t)=b_1t^2+b_2t+b_3,\\
&\Omega^1(X,t,x)=b_1tx+a_2x+\beta_1tX+\beta_2t+\beta_3X+\beta_4,\\
&\Omega^2(t,\theta)=-4b_1t\theta+\mu_1\theta, \qquad (17)
\end{aligned}$$

where $c_1, c_2, b_1, b_2, b_3, a_2, \beta_1, \beta_2, \beta_3, \beta_4, \mu_1$ are arbitrary conctants, thus for the time being, we have obtained a 11-parameter group of infinitesomal transformations. It is important to warn the reader that the parameters in (17) are not related necessarily with the corresponding ones of (2). Furthermore, it is noted that the number of the parameters will reduce as it will be apparent in next section.
The free energy function is constrained to have the form

$$F(p,\theta) = f(p)\theta^2 + \phi(p), \tag{18}$$

where f and ϕ are arbitrary functions and the conductivity function should fulfil the equations

$$(k'/k)\omega^1 = c, \tag{19}$$

where c is an arbitrary constant and

$$(k'/k)\omega^1 + 3\dot{\omega}^2 - 2\omega^{1'} + \frac{\partial\Omega^2}{\partial\theta} - 2\frac{\partial\Omega^1}{\partial X} = 0. \tag{20}$$

3 The Isovector Field

In this section, after an exchausting elaboration of eqs (11)—(20), we will give the main result, that is the isovector field for homogeneous thermoelasticity. The way we follow to elaborate them demands that equations (11)—(15) admit isovector field (17). This will result in new equations to which functions f and ϕ should obey, which in turn will provide new relationships for the parameters and the isovector itself. Before we apply this procedure, we insert (17) into relationship (20) to obtain

$$c = -3b_2 + 2c_1 - \mu_1 + 2a_2. \tag{21}$$

Next, inserting isovector field (17) into eq. (11) we get

$$\begin{gathered}(-c_1 + 2(2b_1t + b_2) - (b_1t + a_2) + (-4b_1t + \mu_1)2\theta f_p + \\ (-4b_1t + \mu_1)2\theta f_p + (\beta_1 t + \beta_3 - pc_1 + p(b_1t + a_2)2\theta f_p = 0,\end{gathered}$$

from which we obtain

$$(-c_1 + 2b_2 - a_2 + 2\mu_1)f_p + (\beta_3 - p(c_1 - a_2))f_{pp} = 0, \tag{22}$$
$$-5b_1 f_p + (\beta_1 + pb_1)f_{pp} = 0. \tag{23}$$

Doing the same with eq. (12) will give nothing because, provided eqs. (17), it (eq. (12)) is fulfilled identically.

Equation (13) with the help of relationship (21) will give

$$(4b_2 - 5c_1 + 2\mu_1 - 4a_2)f + (\beta_3 - p(c_1 - a_2))f_p = 0, \tag{24}$$

$$-4b_1 f + (\beta_1 + pb_1)f_p = 0. \tag{25}$$

The same procedure applied on eq. (14) will give eq. (25) too.

Following the same line of argument, eq. (15) in turn will become

$$\begin{aligned} \theta t^2[-4b_1 f_{pp} + (\beta_1 + pb_1)f_{ppp}] + t[4b_1\phi_{pp} + (\beta_1 + pb_1)f_{ppp}] + \\ \theta^2[(-2c_1 + 2b_2 + 2\mu_1)f_{pp} + [\beta_3 - p(c_1 - a_2)]f_{ppp}] + \\ [(-2c_1 + 2b_2)\phi_{pp} + [\beta_3 - p(c_1 - a_2)]\phi_{ppp}] = 0, \end{aligned}$$

from which we obtain

$$(-2c_1 + 2b_2 + 2\mu_1)f_{pp} + [\beta_3 - p(c_1 - a_2)]f_{ppp} = 0, \tag{26}$$

$$-4b_1 f_{pp} + (\beta_1 + pb_1)f_{ppp} = 0, \tag{27}$$

$$4b_1\phi_{pp} + (\beta_1 + pb_1)f_{ppp} = 0, \tag{28}$$

$$(-2c_1 + 2b_2)\phi_{pp} + [\beta_3 - p(c_1 - a_2)]\phi_{ppp} = 0. \tag{29}$$

The last one of the equations we treat, namely eq. (16), does not have any interest because it provides eqs. (22)—(23) which we have already taken from eq. (11). To clear up the situation we remark that all the equations (22)—(27) concern function f, so they should be valid simoultaneously. Demanding this we can obtain new relationships between the parameters, which will lead to the reduction of parameters number that is the modification of the isovector field itself.

It is easy for one to see that equations (22), (23) and (26), (27) respectively are compatible, by means that the former provides the latter by a simple derivation. Hence, we treat only eqs. (22), (23) which in turn should be compatible to eqs. (24)—(25). Differentiating eq. (25) we take

$$-3b_1 f_p + (\beta_1 + pb_1)f_p = 0. \tag{30}$$

Comparing this with eq. (23), we obtain

$$b_1 = 0, \tag{31}$$

and

$$\beta_1 = 0, \quad \text{or} \quad f \quad \text{constant}. \tag{32}$$

Obviously, choosing f to be constant, we conclude a free energy function of the form

$$F(p,\theta) = \lambda\theta^2 + \phi(p), \quad \lambda \quad \text{constant}$$

which does not provide coupled thermoelasticity, hence we proceed adopting the first choice, i.e.,

$$\beta_1 = 0 \tag{33}$$

Differentiating now eq. (24), we obtain

$$(4b_2 - 6c_1 + 2\mu_1 - 3a_2)f_p + (\beta_3 - p(c_1 - a_2))f_{pp} = 0,$$

which after comparison with eq. (22) will give

$$c_1 = \frac{2}{5}(b_2 - a_2). \tag{34}$$

After obtaining the relationships (31), (33) and (34) betwwen the parameters, it remains a unique differential equation that f should fulfil

$$(4b_2 + \mu_1 - a_2)f + (\beta_3 - \frac{p}{5}(2b_2 - 7a_2))f_p = 0. \tag{35}$$

The same is true for function ϕ. Inserting eqs. (30), (32) and (33) into eqs. (28) and (29), the former is fulfilled identically and the latter takes the form

$$\frac{2}{5}(3b_2 + 2a_2)\phi_{pp} + (\beta_3 - \frac{p}{5}(2b_2 - 7a_2))\phi_{ppp} = 0. \tag{36}$$

It is now apparent that the only meaningful choice regarding proposition (32) is $\beta_1 = 0$. Otherwise, we will necessarily conclude that $\phi =$ constant which does not make any sense for thermoelasticity. Actually, there will rise a free energy function depending only on temparature field θ, thus appropriate for a linear heat conduction theory for rigid media.

Let us return to the differential equation (19) governing the behavior of the conductivity function k. We recall this equation as it is

$$(k^{'}/k)\omega^1 = c, \tag{37}$$

noting that c is not any more an arbitrary constant, but it is linked with the parameters of the symmetry group through the relationship

$$c = -\frac{11}{5}b_2 + \frac{6}{5}a_2 - \mu_1. \tag{38}$$

Equation (38) directly rises from eqs. (21) and (34).

We are passing now to the isovector field in which we enter all information about the parameters, inserting eqs. (31), (33) and (34) into eq. (17)

$$\begin{aligned}
&\omega^1(X) = \frac{2}{5}(b_2 - a_2)X + c_2, \\
&\omega^2(t) = b_2 t + b_3, \\
&\Omega^1(X,t,x) = a_2 x + \beta_2 t + \beta_3 X + \beta_4, \\
&\Omega^2(t,\theta) = \mu_1 \theta.
\end{aligned} \tag{39}$$

Thus, we finally obtain a 8-parameter group of transformations which we will examine in detail in the next section. For the time being, summarizing our main conclusion we can claim:

The symmetry group admitted by the system of one-dimensional, non-linear, homogeneous thermoelasticity (7), (8), is given by (39) provided that the free energy function is of the form

$$F(p,\theta) = f(p)\theta^2 + \phi(p), \tag{40}$$

where the functions f and ϕ fulfil the differential equations (35) and (36), respectively and the heat conductivity function k is governed by the differential equation (36) - 37).

4 Special Cases of Symmetries

The results presented in last section continue to be so general that they can not let us scrutinize particular cases of probably practical interest. This is due to the fact that our main equations (7) and (8) are not a sole system; actually they make up a class of equations, depended upon free energy function F. Hence, for every choice of F we take a seperate member of the class. To talk about symmetries one must first talk about the form of function F. This is already apparent due to the fact that our main result on admissible symmetries (39) depends on the class of functions F having the form (40). To proceed further one need to have particular F, or to put further constraints on the free energy function. This is exactly our next step.

4.1 The funcion f is arbitrary

Letting full arbitrariness to f means the differential equation (35) is valid for every f, hence the coefficients of the equation should be

$$\beta_3 = 0, \quad b_2 = \frac{7}{2}a_2, \quad \mu_1 = -\frac{5}{2}a_2. \tag{41}$$

After eqs. (41), the isovector field (39) becomes

$$\begin{aligned} \omega^1(X) &= a_2 X + c_2, \\ \omega^2(t) &= \frac{7}{2}a_2 t + b_3, \\ \Omega^1(t,x) &= a_2 x + \beta_2 t + \beta_4, \\ \Omega^2(\theta) &= -\frac{5}{2}a_2\theta \end{aligned} \tag{42}$$

and the differential equations for ϕ and k become

$$\frac{23}{5}a_2\phi_{pp} = 0 \Rightarrow \phi_{pp} = 0, \quad \text{for} \quad a_2 \neq 0$$

and

$$(k'/k)\omega^1 = -4a_2.$$

Hence we obtain

$$k(X) = (a_2 X + c_2)^{-4}, \quad \phi(p) = \phi_1 p + \phi_2, \tag{43}$$

where c_2, ϕ_1 and ϕ_2 are arbitrary constants. Hence, for the case under study our initial system (7)—(8) is constrained to have the form

$$f''(p)\theta^2\frac{\partial^2 x}{\partial X^2} + 2f'(p)\theta\frac{\partial\theta}{\partial X} - \rho_0\frac{\partial^2 x}{\partial t^2} = 0, \tag{44}$$

$$k'(X)\frac{\partial\theta}{\partial X} + k(x)\frac{\partial^2 x}{\partial X^2} + 2f(p)\theta\frac{\partial\theta}{\partial t} + 2f'(p)\theta^2\frac{\partial^2 x}{\partial X\partial t} = 0. \tag{45}$$

We summarize what we have found for this particular case in the following statement:

If the differential equations (44)—(45) admit the symmetries given by isovector field (42), for arbitrary f then the function k will be necessarily of the form (43a).

Looking at the isovector field (42) we can recognize that the parameter a_2 gives the scaling (symmetry) and c_2, b_3 and β_4 are related with translations with respect to X, t and x, respectively. It is worthwhile to further examine the symmetry of scalings, thus to set $a_2 \neq 0$ and $c_2 = b_3 = \beta_2 = \beta_4 = 0$. In other words we examine the particular infinitesimal generator

$$V = X\frac{\partial}{\partial X} + \frac{7}{2}t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{5}{2}\theta\frac{\partial}{\partial \theta},$$

or the particular Lie group of finite transformations of scaloing type

$$X^* = e^{\epsilon}X, \quad t^* = e^{\frac{7}{2}\epsilon}t, \quad x^* = e^{\epsilon}x, \quad \theta^* = e^{\frac{7}{2}\epsilon}\theta. \tag{46}$$

The previous analysis secures that the transformation group (46) is admitted by PDE (44)—(45) provided the conductivity function k is of the particular form

$$k(X) = CX^{-4}. \tag{47}$$

4.2 The function ϕ is arbtrirary

We let now eq. (36) be valid for every ϕ which results

$$b_2 = -\frac{2}{3}a_2, \quad b_2 = \frac{7}{2}a_2, \quad \beta_3 = 0. \tag{48}$$

Relations (48) make sense only if $a_2 = b_2 = 0$. Hence the isovector field (39) becomes

$$\begin{aligned} \omega^1(X) &= c_2, \\ \omega^2(t) &= b_3, \\ \Omega^1(t,x) &= \beta_2 t + \beta_4, \\ \Omega^2(\theta) &= \mu_1\theta \end{aligned} \tag{49}$$

and eq. (35) takes the form

$$2\mu_1 f = 0.$$

If we want to keep the symmetry related to the parameter μ_1, we must necessarily consider $f = 0$, which in turn means that

$$F = F(p) = \phi(p), \quad \phi \quad \text{arbitrary},$$

which, certainly, does not lead to any kind of thermoelasticity. In order to have thermoelasticity, we must put $\mu_1 = 0$, hence f *is an arbitrary function* and the isovector field becomes

$$\begin{aligned}\omega^1 &= c_2, \\ \omega^2 &= b_3, \\ \Omega^1(t) &= \beta_2 t + \beta_4, \\ \Omega^2 &= 0\end{aligned} \tag{50}$$

and the free energy funcion will take the form

$$F(p, \theta) = f(p)\theta^2 + \phi(p), \tag{51}$$

where f and ϕ are arbitrary functions.

We examine now the specific symmetry corresponding to the parameter $\beta_2 \neq 0$, which arises within the case 4.1 as well. (It is worthwhile to examine whether the case 4.1 for $\beta_2 \neq 0$ will give us the arbitrariness of ϕ which we enjoy in the present case). In other words we discuss about the symmetry

$$X^* = X, \quad t^* = t, \quad x^* = x + t\epsilon, \quad \theta^* = \theta. \tag{52}$$

Recalling now that equation $k(X)$ should obey

$$(k^{'}/k)\omega^1 = -4a_2,$$

it is easy to conclude that $k' = 0$, thus the function $k(X)$ *is becoming a simple constant.* After that the field equations of thermoelasticity (i.e., eqs. (7)—(8) will take the form:

$$[f''(p)\theta^2 + \phi''(p)]\frac{\partial^2 x}{\partial X^2} + 2f'(p)\theta\frac{\partial \theta}{\partial X} - \rho_0\frac{\partial^2 x}{\partial t^2} = 0, \tag{53}$$

$$k\frac{\partial^2 x}{\partial X^2} + 2f(p)\theta\frac{\partial \theta}{\partial t} + 2f'(p)\theta^2\frac{\partial^2 x}{\partial X \partial t} = 0. \tag{54}$$

Concluding, we can claim that *the symmetry group given by eqs. (52) is the unique symmetry that is admitted by full homogeneous (k, ρ_0 constants) non-linear thermoelastic materials governed by eqs. (53)—(54).*

4.3 The function k is arbitrary

In order the function k to be an arbitrary one, i.e., every function k to satisfy the differential equation (37), we must put

$$\omega_1 \equiv 0, \quad c = 0, \tag{55}$$

from which we conclude straightforwordly

$$b_2 = a_2, \quad c_2 = 0. \tag{56}$$

Furthermore, in virtue of eq. (38) we obtain

$$\mu_1 = -a_2. \tag{57}$$

So, after eqs. (56)—(57) we obtain for the comonents of the isovector field:

$$\begin{aligned} &\omega^1 = 0, \\ &\omega^2(t) = a_2 t + b_3, \\ &\Omega^1(X,t,x) = a_2 x + \beta_2 t + \beta_3 X + \beta_4, \\ &\Omega^2(\theta) = -a_2\theta. \end{aligned} \tag{58}$$

Coming back now, to eqs. (35)—(36) which for the case under study they take the specific form

$$\begin{aligned} 2a_2 f - (\beta_3 + pa_2) f_p = 0, \\ 2a_2 \phi_{pp} + (\beta_3 + pa_2)\phi_{ppp} = 0. \end{aligned} \tag{59}$$

After that, the field equations (7)-(8) for the four-parameter symmetry group (58) take the form

$$[f''(p)\theta^2 + \phi''(p)]\frac{\partial^2 x}{\partial X^2} + 2f'(p)\theta\frac{\partial \theta}{\partial X} - \rho_0\frac{\partial^2 x}{\partial t^2} = 0, \tag{60}$$

$$k'(X)\frac{\partial \theta}{\partial X} + k(X)\frac{\partial^2 \theta}{\partial X^2} + 2f(p)\theta\frac{\partial \theta}{\partial t} + 2f'(p)\theta^2\frac{\partial^2 x}{\partial X \partial t} = 0. \tag{61}$$

The most interesting symmetry for the case under discussion seems to be the corresponding one to the parameter a_2:

$$V = \frac{\partial}{\partial X} + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \theta\frac{\partial}{\partial \theta},$$

or in the form of a transformation group

$$X^* = X, \quad t^* = e^{\epsilon}t, \quad x^* = e^{\epsilon}x, \quad \theta^* = e^{-\epsilon}\theta. \tag{62}$$

For this particular symmetry, the differential equations (59)—(60) take the form

$$\begin{aligned} 2f - pf_p = 0, \\ 2\phi_{pp} + p\phi_{ppp} = 0, \end{aligned} \tag{63}$$

Thus the functions f and ϕ should have the form

$$f(p) = C_1 p^2, \quad \phi(p) = C_2 \ln p + C_3 p + C_4, \tag{64}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

5 Similarity Solutions

The next question is whether the symmetries we have found in last section, give any of the so-called group invariant solutions. We remind here that an invariant solution for a group of transformations admitted by the field equations, is nothing but a solution of the field equations which moreover is invariant under this group. The well-known similarity solutions are invariant solutions corresponding to the particular case of a scaling group. That means that a solution (x, θ) of the field equations (7)—(8) is an invariant one, for a given symmetry in the form of eq. (9), with $\omega^i \neq 0, i = 1, 2$, if they fulfil the differential equations

$$\begin{aligned} \omega^1 \frac{\partial x}{\partial X} + \omega^2 \frac{\partial x}{\partial t} &= \Omega^1, \\ \omega^1 \frac{\partial \theta}{\partial X} + \omega^2 \frac{\partial \theta}{\partial t} &= \Omega^2. \end{aligned} \tag{65}$$

So, we have to check all symmetries deriving in last section under this requirement

• *Symmetry given by eqs. (46).*

In this case the PDEs (65) take the form

$$X \frac{\partial x}{\partial X} + \frac{7}{2} t \frac{\partial x}{\partial t} = x, \tag{66}$$

$$X \frac{\partial \theta}{\partial X} + \frac{7}{2} t \frac{\partial \theta}{\partial t} = -\frac{5}{2} \theta. \tag{67}$$

Their solutions will be

$$x(X, \xi) = u(\xi) X, \quad \theta(x, \xi) = v(\xi) X^{-\frac{5}{2}}, \tag{68}$$

where $\xi = X t^{-\frac{2}{7}}$ is the similarity variable. Thus the function given by eqs. (68) will be the similarity solution of field equations (44)—(45). In order to check this, we have to carry out some calculations:

$$p = \frac{\partial x}{\partial X} = u'(\xi) \frac{\partial \xi}{\partial X} X + u(\xi) = u'(\xi) t^{-\frac{2}{7}} X + u(\xi) \Rightarrow$$

$$p = \xi u'(\xi) + u(\xi). \tag{69}$$

In the same manner we obtain

$$\frac{\partial^2 x}{\partial X^2} = t^{-\frac{2}{7}}(2u'(\xi) + \xi u''(\xi)), \tag{70}$$

$$\frac{\partial x}{\partial t} = -\frac{2}{7}u'(\xi)\xi^2 t^{-\frac{5}{7}}, \tag{71}$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{4}{49}u''(\xi)\xi^3 t^{-\frac{12}{7}} + \frac{18}{49}u'(\xi)\xi^2 t^{-\frac{12}{7}}, \tag{72}$$

$$\frac{\partial^2 x}{\partial X \partial t} = -\frac{2}{7}u''(\xi)\xi^2 t^{-1} - \frac{4}{7}u'(\xi)\xi t^{-1}, \tag{73}$$

$$\frac{\partial \theta}{\partial X} = v'(\xi)\xi X^{-\frac{7}{2}} - \frac{5}{2}v(\xi)X^{-\frac{7}{2}}, \tag{74}$$

$$\frac{\partial^2 \theta}{\partial X^2} = [v''(\xi)\xi^2 - 5v'(\xi)\xi + \frac{35}{4}v(\xi)]X^{-\frac{9}{2}}, \tag{75}$$

$$\frac{\partial \theta}{\partial t} = -\frac{2}{7}v'(\xi)\xi t^{-1}. \tag{76}$$

Substituting now eqs. (47) and (67)—(76) into eqs. (44)—(45), we obtain

$$f''(P)v^2(2\xi u' + \xi^2 u'') + f'(p)(2\xi v v' - 5v^2) - \rho_0(\frac{4}{49}\xi^9 u'' + \frac{18}{49}\xi^8 u') = 0, \tag{77}$$

$$C(\xi^2 v'' - 9\xi v' + \frac{75}{4}v) - \frac{4}{7}f(p)\xi^{\frac{9}{2}} v v' - \frac{4}{7}f'(p)v^2(\xi^{\frac{11}{2}}u'' + 2\xi^{\frac{9}{2}}u') = 0. \tag{78}$$

It is worthwhile to note that the above system consists of highly non-linear but *ordinary* differential equations as we expected to. This is an indirect confirmation that all the previous analysis was carried out correctly.

- *Symmetry given by eqs. (62)*

In this case, eqs. (65) will take the form

$$t\frac{\partial x}{\partial t} = x, \quad t\frac{\partial \theta}{\partial t} = -\theta. \tag{79}$$

Hence, the similarity solutions corresponding to the symmetry (62) must be of the form

$$x(X,t) = u(X)t, \quad \theta(x,t) = v(X)t^{-1}, \tag{80}$$

where u and v are arbitrary functions of X. To be sure that eqs. (80) are indeed similarity solutions we must check whether they reduce the number of the indepedent variableds of the system (60)—(61). Actually, we will check whether or not, eqs. (80) will transform the aforementioned system to a system of ordinary differential equations. Indeed, inserting eqs. (64) into PDEs (60)—(61) we obtain

$$C_1\theta^2\frac{\partial^2 x}{\partial X^2}+2C_1\theta\frac{\partial x}{\partial X}\frac{\partial\theta}{\partial X}-C_2(\frac{\partial x}{\partial X})^{-2}\frac{\partial^2 x}{\partial X^2}-\rho_0\frac{\partial^2 x}{\partial t^2}=0, \tag{81}$$

$$k'(X)\frac{\partial\theta}{\partial X}+k(X)\frac{\partial^2\theta}{\partial X^2}+C_1\theta\frac{\partial\theta}{\partial t}(\frac{\partial x}{\partial X})^2+2C_1\theta^2\frac{\partial x}{\partial X}\frac{\partial^2 x}{\partial X\partial t}=0. \tag{82}$$

After that, we carry out the following calculations

$$\frac{\partial x}{\partial X}=u'(X)t,\quad \frac{\partial^2 x}{\partial X^2}=u''(X)t,\quad \frac{\partial x}{\partial t}=u(X),$$
$$\frac{\partial^2 x}{\partial t^2}=0,\quad \frac{\partial^2 x}{\partial X\partial t}=u'(X) \tag{83}$$

and

$$\frac{\partial\theta}{\partial X}=v'(X)t^{-1},\quad \frac{\partial^2\theta}{\partial X^2}=v''(X)t^{-1},\quad \frac{\partial\theta}{\partial t}=-v(X)t^{-2}. \tag{84}$$

Thus, in our last step we have just to insert eqs. (83)–(84) into eqs. (81)–(82) to obtain the following system of ordinary differential equations

$$C_1v^2u''+2C_1vv'u'-C_2u'u''=0, \tag{85}$$
$$k'(X)v'+k(x)v''-C_1v^2u'+2C_1v^2u'^2=0. \tag{86}$$

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A Appendix: The Derivation of Eqs. (17)—(20)

We give here the technical details for the derivation of eqs.(17)—(20). Summarizing the information about isovector field at hand, namely equations (16), we can write

$$\begin{aligned}\omega^2(t) &= b_1t^2 + b_2t + b_3,\\ \Omega^1(X,t,x) &= (b_1t + a_2)x + \beta(X,t),\\ \Omega^2(X,t,\theta) &= [\lambda(X) + \mu(t)]\theta + \gamma(X,t),\end{aligned} \tag{87}$$

where b_1, b_2, b_3 and a_2 are arbitrary constants, β, γ, μ are arbitrary functions and

$$\lambda(X) = \frac{1}{2}\int k(X)dX$$

If we differentiate now eqs. (10) and (11) with respect to X and θ respectively and after that subtract them from each other we can obtain by virtue of eqs. (16), for $F_{p\theta} \neq 0$

$$\omega^{1''} = 0 \Rightarrow \omega^1 = c_1X + c_2. \tag{88}$$

Hence, up to this point we have proved the form of the first two components of the isovector field (17).

Differentiating eq. (14) with respect to θ, multiplying by θ, and then differentiating (15) with respect to p and finally subtracting from each other we take

$$\Big[\big[(k^{'}/k)\omega^1 + 3\dot{\omega^2} - 2\omega^{1'} + 2\frac{\partial\Omega^2}{\partial\theta} - 2\frac{\partial\Omega^1}{\partial x}\big]\theta - \Omega^2\Big]\theta F_{\theta pp} = 0.$$

Thus for $F_{\theta pp} \neq 0$, we obtain

$$(k^{'}/k)\omega^1 + 3\dot{\omega^2} - 2\omega^{1'} + \frac{\partial\Omega^2}{\partial\theta} - 2\frac{\partial\Omega^1}{\partial x} = 0, \tag{89}$$

which is the required eq. (20).

In what follows we elaborate carefully eq. (89); first we differentiate with respect to t and easily obtain

$$\mu(t) = -4b_1t + \mu_1, \tag{90}$$

where μ_1 is an arbitrary constant. Next differentiating with respect to X, we obtain

$$(k^{'}/k)\omega^1 = c. \tag{91}$$

Therefore, eq. (19) has derived. Moreover, with the aid of eq. (16) it gives

$$\frac{\partial^2 \Omega^2}{\partial X \partial \theta} = 0. \tag{92}$$

Inserting now eq. (89) into eq. (12) it becomes comparable to eq. (10). After proper differentiation this comparison results

$$\frac{\partial^2 \Omega^2}{\partial t^2} = 0. \tag{93}$$

Coming back to eqs. (10) and (11), differentiating them twice with respect to t , we can obtain

$$\frac{\partial^3 \Omega^1}{\partial X \partial t^2} = 0, \quad \frac{\partial^4 \Omega^1}{\partial t^4} = 0. \tag{94}$$

In the same line of argument, with the aid of eq. (89) we can make eqs. (10) and (15) to be comparable with each other. Thus for $F_{\theta p} \neq 0$ we obtain

$$\frac{\partial \Omega^2}{\partial \theta} - \Omega^2 = 0 \Rightarrow \Omega^2(t, \theta) = (-4b_1 t + \mu_1)\theta \tag{95}$$

The last step was obtained by virtue of eq. (92). After the last relation, we come back to eqs. (10) and (11) once more, differentiate with respect to X and t respectively to obtain

$$\frac{\partial^2 \Omega^1}{\partial X^2} = 0, \quad \frac{\partial^2 \Omega^1}{\partial t^2} = 0. \tag{96}$$

With eqs. (94) and (96) at hand the function β takes the form

$$\beta(X, t) = \beta_1 t X + \beta_2 t + \beta_3 X + \beta_4, \tag{97}$$

where $\beta_1, \beta_2, \beta_3$ and β_4 are arbitrary constants. After eqs. (95) and (97) the form of the remaining components of the isovector field (17) has been proved too.

Last, we apply the same way of elaboration to eqs. (13) and (15); that is we differentiate with respect to θ the former and with respect to t the latter and substitute each other. the result of this manipulation is

$$\theta F_{\theta\theta} - F_\theta = 0 \Rightarrow F(p, \theta) = f(p)\theta^2 + \phi(p), \tag{98}$$

where f and ϕ are arbitrary functions. Hence, eq. (18) has derived, too.